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Integrable Rosochatius deformations of higher-order constrained flows and the soliton hierarchy with self-consistent sources

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Abstract

We propose a systematic method for generalizing the integrable Rosochatius deformations for finite-dimensional integrable Hamiltonian systems to integrable Rosochatius deformations for infinite-dimensional integrable equations. An infinite number of the integrable Rosochatius deformed higher-order constrained flows of some soliton hierarchies, which includes the generalized integrable Hénon–Heiles system, and the integrable Rosochatius deformations of the KdV hierarchy with self-consistent sources, of the AKNS hierarchy with self-consistent sources and of the mKdV hierarchy with self-consistent sources as well as their Lax representations are presented.

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1. Introduction

Rosochatius found that it would still keep the integrability to add a potential of the sum of inverse squares of the coordinates to that of the Neumann system [1, 2]. The deformed system is called the Neumann–Rosochatius system. Wojciechowski obtained an analogy system for the Garnier system as a stationary KdV flow in 1985 [3, 4]. In 1997, Kubo *et al* [5] constructed the analogy system for the Jacobi system [6] and the geodesic flow equation on the ellipsoid based upon the Deift technique and a theorem that the Gauss map transforms the Neumann system to the Jacobi system [7, 8]. All of these systems have the same character that they are integrable Hamiltonian systems containing N arbitrary parameters and the original finite-dimensional integrable Hamiltonian systems are recovered when all these parameters vanish. In fact, these systems are a sort of integrable deformation of the corresponding integrable Hamiltonian systems, which are called integrable Rosochatius deformations. The resulting systems are called the Rosochatius-type integrable systems. These systems have important physical applications. For example, the Neumann–Rosochatius system can be used to describe the dynamics of rotating closed string solutions in $AdS_5 \times S^5$ and the membranes on $AdS_4 \times S^7$ [9–17]. The Garnier–Rosochatius system can be used to solve the multicomponent coupled nonlinear Schrödinger equation [3, 4, 18]. It is not difficult to see that each of the above systems has its own origin. In [19], Zhou generalizes the Rosochatius method for studying the integrable Rosochatius deformations of some explicit constrained flows of soliton equations.

However, so far the Rosochatius deformations are limited to few finite-dimensional integrable Hamiltonian systems (FDIHSs). It is natural to ask whether there exist integrable Rosochatius deformations for infinite-dimensional integrable equations. The main purpose of this paper is to generalize the Rosochatius deformation from FDIHSs to infinite-dimensional integrable equations. We will investigate the integrable Rosochatius deformations first for an infinite number of higher-order constrained flows of some soliton equations, then for some soliton hierarchies with self-consistent sources.

In recent years the constrained flows of soliton equations obtained from the symmetry reduction of soliton equations, which can be transformed into a FDIHS, attracted a lot of attention [20–33]. Many well-known FDIHSs are recovered by means of constrained flows. Furthermore, a soliton equation can be factorized into two commuting constrained flows, which provides an effective way of solving the soliton equations by solving the constrained flows.

To make the paper self-contained, we first briefly recall the higher-order constrained flows of the soliton hierarchy and the soliton hierarchy with self-consistent sources. Consider a hierarchy of soliton equations which can be formulated as infinite-dimensional Hamiltonian systems

$$u_{t_n} = J\frac{\delta H_n}{\delta u}. \tag{1}$$

The auxiliary linear problems associated with (1) are given by

$$\phi_x = U(\lambda, u)\phi, \qquad \phi = (\phi_1, \phi_2)^T \tag{2a}$$

$$\phi_{t_n} = V^{(n)}(\lambda, u)\phi. \tag{2b}$$

Then the higher-order constrained flows of (1) consist of the equations obtained from the spectral problem (2*a*) for N distinct λ_j and the restriction of the variational derivatives for the conserved quantities H_n and λ_j [23]:

$$J\left[\frac{\delta H_n}{\delta u} + \sum_{j=1}^{N}\frac{\delta\lambda_j}{\delta u}\right] = 0, \tag{3a}$$

$$\phi_{j,x} = U(\lambda_j, u)\phi_j, \qquad \phi_j = (\phi_{1j}, \phi_{2j})^T, \qquad j = 1, 2, \ldots, N. \tag{3b}$$

By introduction of the so-called Jacobi–Ostrogradsky coordinates [34], (3) can be transformed into a FDIHS. The Lax representation of (3) can be deduced from the adjoint representation of (2) [24]

$$N_x^{(n)} = [U, N^{(n)}]. \tag{4}$$

Mainly, $N^{(n)}$ has the following forms:

$$N^{(n)}(\lambda) = V^{(n)}(\lambda) + N_0 = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & -A(\lambda) \end{pmatrix},$$

where

$$N_0=\sum_{j=1}^{N}\frac{1}{\lambda-\lambda_j}\begin{pmatrix}\phi_{1j}\phi_{2j} & -\phi_{1j}^2\\ \phi_{2j}^2 & -\phi_{1j}\phi_{2j}\end{pmatrix},\quad \text{or}\quad N_0=\sum_{j=1}^{N}\frac{1}{\lambda^2-\lambda_j^2}\begin{pmatrix}\lambda\phi_{1j}\phi_{2j} & -\lambda_j\phi_{1j}^2\\ \lambda_j\phi_{2j}^2 & -\lambda\phi_{1j}\phi_{2j}\end{pmatrix}.$$

The soliton hierarchy with self-consistent sources is defined by [25, 26]

$$u_{t_n}=J\left[\frac{\delta H_n}{\delta u}+\sum_{j=1}^{N}\frac{\delta\lambda_j}{\delta u}\right],\tag{5a}$$

$$\phi_{j,x}=U(\lambda_j,u)\phi_j,\qquad j=1,2,\ldots,N.\tag{5b}$$

Since the higher-order constrained flows (3) are just the stationary equations of (5), the zero-curvature representation for (5) can be induced from (4) as follows:

$$U_{t_n}-N_x^{(n)}+[U,N^{(n)}]=0,\tag{6}$$

which implies that (5) is Lax integrable. In fact, (5) can also be formulated as an infinite-dimensional integrable Hamiltonian system with a t-type Hamiltonian operator by taking t as the 'spatial' variable and x as the evolution parameter as well as by introducing the Jacobi–Ostrogradsky coordinates [35, 36]. Then we can find the Poisson bracket defined by the t-type Hamiltonian operator, and the conserved density when considering x as the evolution parameter. The t-type bi-Hamiltonian description for the KdV equation with self-consistent sources and for the Jaulent–Miodek equation with self-consistent sources were presented in [35, 36], respectively.

The soliton equations with self-consistent sources have important physical applications; for example, the KdV equation with self-consistent sources describes the interaction of long and short capillary–gravity waves [37–39].

In this paper, in the same way [19], we construct the Rosochatius deformation $\tilde{N}^{(n)}$ of Lax matrix $N^{(n)}$ by replacing ϕ_{2j}^2 in the matrix N_0 with $\phi_{2j}^2+\frac{\mu_j}{\phi_{1j}^2}$; namely the entries of $\tilde{N}^{(n)}$ are given by

$$\tilde{A}(\lambda)=A(\lambda),\qquad \tilde{B}(\lambda)=B(\lambda),\qquad \tilde{C}(\lambda)=C(\lambda)+\sum_{j=1}^{N}\frac{\mu_j}{(\lambda-\lambda_j)\phi_{1j}^2},$$

or

$$\tilde{C}(\lambda)=C(\lambda)+\sum_{j=1}^{N}\frac{\lambda_j\mu_j}{(\lambda^2-\lambda_j^2)\phi_{1j}^2}.\tag{7}$$

Then Lax representation (4) with $N^{(n)}$ replaced by $\tilde{N}^{(n)}$ gives rise to the Rosochatius deformations of (3). The fact that such a substitute keeps the relations of the Poisson brackets of $A(\lambda)$, $B(\lambda)$ and $C(\lambda)$ guarantees the integrability of Rosochatius deformations of (3). In this way we can obtain an infinite number of integrable Rosochatius deformed FDIHSs constructed from the Rosochatius deformed higher-order constrained flows of KdV hierarchy, AKNS hierarchy and mKdV hierarchy, respectively. Among these Rosochatius deformed FDIHSs, it needs to be pointed out that the Rosochatius deformation of the first higher-order constrained flow of KdV hierarchy contains the well-known generalized integrable Hénon–Heiles system, and can be regarded as the integrable multidimensional extension of the Hénon–Heiles system. Then, based on the Rosochatius deformed higher-order constrained flows, the Rosochatius deformations of the soliton hierarchy with self-consistent sources (RDSHSCS) can be constructed through (6) with $N^{(n)}$ replaced by $\tilde{N}^{(n)}$. The integrability of the RDSHSCS can be explained by the fact that the RDSHSCS possess the zero-curvature representation (6) with $N^{(n)}$ replaced by $\tilde{N}^{(n)}$ and their stationary reduction is integrable Rosochatius deformations of (3). In this way, we construct Rosochatius deformations of KdV hierarchy with self-consistent sources (RDKdVHSCS), of AKNS hierarchy with self-consistent sources (RDAKNSHSCS) and of mKdV hierarchy with self-consistent sources (RDmKdVHSCS), as well as their zero-curvature representations. We note that there are two kinds of RDSHSCS. For RDKdVHSCS and RDmKdVHSCS, the Rosochatius formed terms only appear in (5*b*). However for RDAKNSHSCS Rosochatius deformed terms occur in both (5*a*) and (5*b*).

The structure of the paper is as follows. In section 2, we present the Rosochatius deformation of the higher-order constrained flows of KdV hierarchy and RDKdVHSCS as well as their Lax representations. In section 3, we obtain the Rosochatius deformation of higher-order constrained flows of AKNS hierarchy and RDAKNSHSCS as well as their Lax representations. In section 4, we obtain the Rosochatius deformation of higher-order constrained flows of mKdV hierarchy and RDmKdVHSCS and their Lax representations. In section 5, a conclusion is made.

2. The Rosochatius deformed KdV hierarchy with self-consistent sources

Consider the Schrödinger equation [40]

$$\phi_{1xx} + (\lambda + u)\phi_1 = 0, \tag{8}$$

which can be written in the matrix form

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_x = U\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \qquad U = \begin{pmatrix}0 & 1\\ -\lambda - u & 0\end{pmatrix}. \tag{9}$$

The adjoint representation of (9) reads

$$V_x = [U, V]. \tag{10}$$

Set

$$V = \sum_{i=1}^{\infty}\begin{pmatrix}a_i & b_i\\ c_i & -a_i\end{pmatrix}\lambda^{-i}. \tag{11}$$

Solving (10) yields

$$a_k = -\tfrac{1}{2}b_{k,x}, \qquad b_{k+1} = Lb_k = -\tfrac{1}{2}L^{k-1}u, \qquad c_k = -\tfrac{1}{2}b_{k,xx} - b_{k+1} - b_k u,$$

$$\begin{aligned}&a_0 = b_0 = 0, \qquad c_0 = -1, \qquad a_1 = 0, \qquad b_1 = 1, \qquad c_1 = -\tfrac{1}{2}u,\\ &a_2 = \tfrac{1}{4}u_x, \qquad b_2 = -\tfrac{1}{2}u, \qquad c_2 = \tfrac{1}{8}(u_{xx} + u^2), \qquad b_3 = \tfrac{1}{8}(u_{xx} + 3u^2), \ldots\end{aligned} \tag{12}$$

where $L = -\frac{1}{4}\partial^2 - u + \frac{1}{2}\partial^{-1}u_x$, $\partial = \frac{\partial}{\partial x}$.

Set

$$V^{(n)} = \sum_{i=1}^{n}\begin{pmatrix}a_i & b_i\\ c_i & -a_i\end{pmatrix}\lambda^{n-i} + \begin{pmatrix}0 & 0\\ b_{n+1} & 0\end{pmatrix}, \tag{13}$$

and take

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_{t_n} = V^{(n)}(u, \lambda)\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}. \tag{14}$$

Then the compatibility of the equations (9) and (14) gives rise to the KdV hierarchy

$$u_{t_n} = -2b_{n+1,x} \equiv \partial \frac{\delta H_n}{\delta u}, \qquad n = 0, 1, \ldots, \tag{15}$$

where $H_n = 4b_{n+2}/2n + 1$. We have

$$\frac{\delta \lambda}{\delta u} = \phi_1^2, \qquad L\phi_1^2 = \lambda \phi_1^2. \tag{16}$$

The higher-order constrained flows of the KdV hierarchy are given by [23]

$$\frac{\delta H_n}{\delta u} - \alpha \sum_{j=1}^{N} \frac{\delta \lambda_j}{\delta u} \equiv -2b_{n+1} - \alpha \sum_{j=1}^{N} \phi_{1j}^2 = 0, \tag{17a}$$

$$\phi_{1j,x} = \phi_{2j}, \qquad \phi_{2j,x} = -(\lambda_j + u)\phi_{1j}, \qquad j = 1, 2, \ldots, N. \tag{17b}$$

According to (12), (16) and (17), we find the Lax representation (4) for (17) with

$$N^{(n)} = \sum_{k=0}^{n} \begin{pmatrix} a_k & b_k \\ c_k & -a_k \end{pmatrix} \lambda^{n-k} + \frac{\alpha}{2} \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} \begin{pmatrix} \phi_{1j}\phi_{2j} & -\phi_{1j}^2 \\ \phi_{2j}^2 & -\phi_{1j}\phi_{2j} \end{pmatrix}. \tag{18}$$

By taking the so-called Jacobi–Ostrogradsky coordinates [34]

$$q_i = u^{(i-1)}, \qquad i = 1, \ldots, n-1,$$
$$p_i = \frac{\delta H_n}{\delta u^{(i)}} = \sum_{l \geqslant 0} (-\partial)^l \frac{\partial H_n}{\partial u^{(i+l)}}$$

and setting

$$\Phi_1 = (\phi_{11}, \phi_{12}, \ldots, \phi_{1N})^T, \qquad \Phi_2 = (\phi_{21}, \phi_{22}, \ldots, \phi_{2N})^T,$$
$$Q = (\phi_{11}, \phi_{12}, \ldots, \phi_{1N}, q_1, \ldots, q_{n-1})^T,$$
$$P = (\phi_{21}, \phi_{22}, \ldots, \phi_{2N}, p_1, \ldots, p_{n-1})^T, \qquad \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N),$$

equation (17) with $\alpha = \frac{2}{4^n}$ can be transformed into a FDIHS [23]

$$Q_x = \frac{\partial H}{\partial P}, \qquad P_x = -\frac{\partial H}{\partial Q}, \tag{19}$$

with

$$H = \sum_{i=1}^{n-1} q_{i,x} p_i - H_n + \frac{1}{2}\langle \Phi_2, \Phi_2 \rangle + \frac{1}{2}\langle \Lambda\Phi_1, \Phi_1 \rangle + \frac{1}{2} q_1 \langle \Phi_1, \Phi_1 \rangle,$$

where $\langle\,\rangle$ denotes the inner production in R^N. For example, (17) for $n = 0$, $\alpha = -2$ gives rise to the Neumann system [20], (17) for $n = 1$, $\alpha = 1$ leads to the Garnier system [3, 20]. When $n = 2$, equation (17) for $\alpha = \frac{1}{8}$ gives the first higher-order constrained flow [23]

$$u_{xx} + 3u^2 = -\frac{1}{2} \sum_{j=1}^{N} \phi_{1j}^2 = -\frac{1}{2} \langle \Phi_1, \Phi_1 \rangle, \tag{20a}$$

$$\phi_{1j,x} = \phi_{2j}, \qquad \phi_{2j,x} = -(\lambda_j + u)\phi_{1j}, \qquad j = 1, 2, \ldots, N. \tag{20b}$$

Let $q_1 = u$, $p_1 = u_x$, (20) becomes a FDIHS (19) with

$$H = \tfrac{1}{2}\langle \Phi_2, \Phi_2 \rangle + \tfrac{1}{2}\langle \Lambda\Phi_1, \Phi_1 \rangle + \tfrac{1}{2} q_1 \langle \Phi_1, \Phi_1 \rangle + \tfrac{1}{2} p_1^2 + q_1^3,$$

and has the Lax representation (4) with the entries of $N^{(2)}$ being given by

$$A(\lambda) = \frac{1}{4}p_1 + \frac{1}{16}\sum_{j=1}^{N}\frac{\phi_{1j}\phi_{2j}}{\lambda-\lambda_j}, \qquad B(\lambda) = \lambda - \frac{1}{2}q_1 - \frac{1}{16}\sum_{j=1}^{N}\frac{\phi_{1j}^2}{\lambda-\lambda_j},$$

$$C(\lambda) = \lambda^2 - \frac{q_1}{2}\lambda - \frac{q_1^2}{4} - \frac{1}{16}\langle\Phi_1,\Phi_1\rangle + \frac{1}{16}\sum_{j=1}^{N}\frac{\phi_{2j}^2}{\lambda-\lambda_j}.$$

With respect to the standard Poisson bracket it is found that

$$\begin{aligned}
&\{A(\lambda),A(\mu)\} = \{B(\lambda),B(\mu)\} = 0, \qquad \{C(\lambda),C(\mu)\} = \frac{A(\lambda)-A(\mu)}{4},\\
&\{A(\lambda),B(\mu)\} = \frac{B(\lambda)-B(\mu)}{8(\lambda-\mu)}, \qquad \{A(\lambda),C(\mu)\} = \frac{C(\lambda)-C(\mu)}{8(\mu-\lambda)} - \frac{B(\lambda)}{8},\\
&\{B(\lambda),C(\mu)\} = \frac{A(\lambda)-A(\mu)}{4(\lambda-\mu)}.
\end{aligned} \tag{21}$$

It follows from (21) that

$$\{A(\lambda)^2 + B(\lambda)C(\lambda), \qquad A(\mu)^2 + B(\mu)C(\mu)\} = 0. \tag{22}$$

When $n = 3$, (17) for $\alpha = \frac{1}{32}$ yields the second higher-order constrained flow [23]

$$u_{xxxx} + 5u_x^2 + 10uu_{xx} + 10u^3 = \tfrac{1}{2}\langle\Phi_1,\Phi_1\rangle, \tag{23a}$$

$$\phi_{1j,x} = \phi_{2j}, \qquad \phi_{2j,x} = -(\lambda_j + u)\phi_{1j}, \qquad j = 1,2,\ldots,N. \tag{23b}$$

Let $q_1 = u$, $q_2 = u_x$, $p_1 = -u_{xxx} - 10uu_x$, $p_2 = u_{xx}$, (23) becomes a FDIHS (19) with $H = \frac{1}{2}\langle\Phi_2,\Phi_2\rangle + \frac{1}{2}\langle\Lambda\Phi_1,\Phi_1\rangle + \frac{1}{2}q_1\langle\Phi_1,\Phi_1\rangle + \frac{1}{2}p_2^2 + q_2p_1 + 5q_1q_2^2 - \frac{5}{2}q_1^4$.
We now consider the Rosochatius deformation $\tilde{N}^{(2)}$ of the Lax matrix $N^{(2)}$ with

$$\tilde{A}(\lambda) = A(\lambda), \qquad \tilde{B}(\lambda) = B(\lambda), \qquad \tilde{C}(\lambda) = C(\lambda) + \frac{1}{16}\sum_{j=1}^{N}\frac{\mu_j}{(\lambda-\lambda_j)\phi_{1j}^2}.$$

It is not difficult to find that $\tilde{A}(\lambda)$, $\tilde{B}(\lambda)$ and $\tilde{C}(\lambda)$ keep the relations of the Poisson brackets (21) and (22).

A direct calculation gives

$$\tilde{A}^2(\lambda) + \tilde{B}(\lambda)\tilde{C}(\lambda) = -\lambda^3 + P_0 + \sum_{j=1}^{N}\frac{P_j}{\lambda-\lambda_j} - \frac{1}{256}\sum_{j=1}^{N}\frac{\mu_j}{(\lambda-\lambda_j)^2}, \tag{24}$$

where

$$\begin{aligned}
P_0 &= \frac{1}{16}\left(\langle\Phi_2,\Phi_2\rangle + \langle\Lambda\Phi_1,\Phi_1\rangle + q_1\langle\Phi_1,\Phi_1\rangle + 2q_1^3 + p_1^2 + \sum_{j=1}^{N}\frac{\mu_j}{\phi_{1j}^2}\right)\\
P_j &= \frac{p_1}{32}\phi_{1j}\phi_{2j} + \frac{1}{16}\left(\lambda_j - \frac{q_1}{2}\right)\left(\phi_{2j}^2 + \frac{\mu_j}{\phi_{1j}^2}\right) + \frac{1}{16}\left(\lambda_j^2 + \frac{q_1}{2}\lambda_j + \frac{1}{16}\langle\Phi_1,\Phi_1\rangle + \frac{q_1^2}{4}\right)\phi_{1j}^2\\
&\quad + \frac{1}{256}\sum_{k\neq j}\frac{1}{\lambda_j-\lambda_k}\left[2\phi_{1j}\phi_{1k}\phi_{2j}\phi_{2k} - \phi_{1j}^2\left(\phi_{2k}^2 + \frac{\mu_k}{\phi_{1k}^2}\right)\right.\\
&\quad \left. - \phi_{1k}^2\left(\phi_{2j}^2 + \frac{\mu_j}{\phi_{1j}^2}\right)\right], \qquad j = 1,\ldots,N.
\end{aligned} \tag{25}$$

Choosing $8P_0 = \tilde{H}$ as a Hamiltonian function, we get the following Hamiltonian system,

$$q_{1x} = p_1, \qquad p_{1x} = -\frac{1}{2}\langle\Phi_1, \Phi_1\rangle - 3q_1^2, \tag{26a}$$

$$\phi_{1jx} = \phi_{2j}, \qquad \phi_{2jx} = -\lambda_j\phi_{1j} - q_1\phi_{1j} + \frac{\mu_j}{\phi_{1j}^3}, \tag{26b}$$

which is the Rosochatius deformation of the first higher-order constrained flow (20). From (4), we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{tr}(\tilde{N}^{(2)}(\lambda))^2 = \frac{\mathrm{d}}{\mathrm{d}x}[\tilde{A}^2(\lambda) + \tilde{B}(\lambda)\tilde{C}(\lambda)] = \operatorname{tr}[U, (\tilde{N}^{(2)}(\lambda))^2] = 0, \tag{27}$$

which implies that $P_0, P_1, \ldots, P_N$ are $N + 1$ independent first integrals of the Hamiltonian system (26). Equality (22) of Poisson bracket for $\tilde{A}(\lambda), \tilde{B}(\lambda)$ and $\tilde{C}(\lambda)$ indicates that $\{P_i, P_j\} = 0, i, j = 0, 1, \ldots, N$. So the Rosochatius deformation (26) of the first higher-order constrained flow (20) is a FDIHS in the Liouville's sense [41].

Remark 1. For $N = 1, \lambda_1 = 0$, (26) yields

$$q_{1xx} = -\frac{1}{2}\phi_1^2 - 3q_1^2,$$

$$\phi_{1xx} = -q_1\phi_1 + \frac{\mu_1}{\phi_1^3},$$

which is the well-known generalized integrable Hénon–Heiles system [42–44]. In fact (26) can be regarded as the integrable multidimensional extension of Hénon–Heiles system.

Similarly, choosing

$$\tilde{H} = \frac{1}{2}\langle\Phi_2, \Phi_2\rangle + \frac{1}{2}\langle\Lambda\Phi_1, \Phi_1\rangle + \frac{1}{2}q_1\langle\Phi_1, \Phi_1\rangle + \frac{1}{2}p_2^2 + q_2p_1 + 5q_1q_2^2 - \frac{5}{2}q_1^4 + \frac{1}{2}\sum_{j=1}^{N}\frac{\mu_j}{\phi_{1j}^2},$$

we get the Rosochatius deformation of the second higher-order constrained flow (23)

$$q_{1x} = q_2, \qquad q_{2x} = p_2, \qquad p_{1x} = 10q_1^3 - 5q_2^2 - \frac{1}{2}\langle\Phi_1, \Phi_1\rangle, \qquad p_{2x} = -10q_1q_2 - p_1, \tag{28a}$$

$$\phi_{1jx} = \phi_{2j}, \qquad \phi_{2jx} = -\lambda_j\phi_{1j} - q_1\phi_{1j} + \frac{\mu_j}{\phi_{1j}^3}, \tag{28b}$$

which, in the same way, can be shown to be a FDIHS.

In general, the integrable Rosochatius deformation of the higher-order constrained flow (19) is generated by the following Hamiltonian function:

$$\tilde{H} = \sum_{i=1}^{n-1} q_{i,x}p_i - H_n + \frac{1}{2}\langle\Phi_2, \Phi_2\rangle + \frac{1}{2}\langle\Lambda\Phi_1, \Phi_1\rangle + \frac{1}{2}q_1\langle\Phi_1, \Phi_1\rangle + \frac{1}{2}\sum_{j=1}^{N}\frac{\mu_j}{\phi_{1j}^2}. \tag{29}$$

The KdV hierarchy with self-consistent sources is defined by [25, 26, 29, 37–39]

$$u_{t_n} = \partial\left[\frac{\delta H_n}{\delta u} - \alpha\sum_{j=1}^{N}\frac{\delta\lambda_j}{\delta u}\right] \equiv \partial\left[-2b_{n+1} - \alpha\sum_{j=1}^{N}\phi_{1j}^2\right], \tag{30a}$$

$$\phi_{1jx} = \phi_{2j}, \qquad \phi_{2j,x} = -(\lambda_j + u)\phi_{1j}, \qquad j = 1, 2, \ldots, N. \tag{30b}$$

Since the higher-order constrained flows (17) are just the stationary equation of the KdV hierarchy with self-consistent sources (30), it is obvious that the zero-curvature representation for the KdV hierarchy with self-consistent sources (30) is given by (6) with [29]

$$N^{(n)} = \sum_{i=1}^{n} \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix} \lambda^{n-i} + \begin{pmatrix} 0 & 0 \\ b_{n+1} + \frac{\alpha}{2}\sum_{j=1}^{N}\phi_{1j}^2 & 0 \end{pmatrix} + \frac{\alpha}{2}\sum_{j=1}^{N}\frac{1}{\lambda-\lambda_j}\begin{pmatrix} \phi_{1j}\phi_{2j} & -\phi_{1j}^2 \\ \phi_{2j}^2 & -\phi_{1j}\phi_{2j} \end{pmatrix}. \tag{31}$$

Equation (30) for $n = 2, \alpha = \frac{1}{8}$ gives rise to the KdV equation with self-consistent sources [29, 37]

$$u_t = -\frac{1}{4}(u_{xxx} + 6uu_x) - \frac{1}{8}\sum_{j=1}^{N}\left(\phi_{1j}^2\right)_x, \tag{32a}$$

$$\phi_{1jx} = \phi_{2j}, \qquad \phi_{2jx} = -(\lambda_j + u)\phi_{1j}, \qquad j = 1, 2, \ldots, N. \tag{32b}$$

Based on (26), the Rosochatius deformation of KdV equation with self-consistent sources is given by

$$u_t = -\frac{1}{4}(u_{xxx} + 6uu_x) - \frac{1}{8}\sum_{j=1}^{N}\left(\phi_{1j}^2\right)_x, \tag{33a}$$

$$\phi_{1jx} = \phi_{2j}, \qquad \phi_{2jx} = -(\lambda_j + u)\phi_{1j} + \frac{\mu_j}{\phi_{1j}^3}, \qquad j = 1, \ldots, N \tag{33b}$$

which has the zero-curvature representation (6) with $N^{(2)}$ given by

$$N^{(2)} = \begin{pmatrix} \frac{u_x}{4} & \lambda - \frac{u}{2} \\ -\lambda^2 - \frac{u}{2}\lambda + \frac{1}{4}u_{xx} + \frac{1}{2}u^2 + \frac{1}{16}\sum_{j=1}^{N}\phi_{1j}^2 & -\frac{u_x}{4} \end{pmatrix} + \frac{1}{16}\sum_{j=1}^{N}\frac{1}{\lambda-\lambda_j}\begin{pmatrix} \phi_{1j}\phi_{2j} & -\phi_{1j}^2 \\ \phi_{2j}^2 + \frac{\mu_j}{\phi_{1j}^2} & -\phi_{1j}\phi_{2j} \end{pmatrix}. \tag{34}$$

Remark 2. The fact that the stationary equation of (33) is a FDIHS (26) and (33) has the zero-curvature representation (6) implies the integrability of the Rosochatius deformation of the KdV equation with the self-consistent source (33).

In general the Rosochatius deformation of the KdV hierarchy with self-consistent sources is given by

$$u_{t_n} = \partial\left[\frac{\delta H_n}{\delta u} - \frac{2}{4^n}\sum_{j=1}^{N}\phi_{1j}^2\right], \tag{35a}$$

$$\phi_{1jx} = \phi_{2j}, \qquad \phi_{2j,x} = -(\lambda_j + u)\phi_{1j} + \frac{\mu_j}{\phi_{1j}^3}, \qquad j = 1, 2, \ldots, N. \tag{35b}$$

which has the zero-curvature representation (6) with $N^{(n)}$ given by

$$N^{(n)} = \sum_{i=1}^{n} \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix} \lambda^{n-i} + \begin{pmatrix} 0 & 0 \\ b_{n+1} + \frac{1}{4^n}\sum_{j=1}^{N}\phi_{1j}^2 & 0 \end{pmatrix} + \frac{1}{4^n}\sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} \begin{pmatrix} \phi_{1j}\phi_{2j} & -\phi_{1j}^2 \\ \phi_{2j}^2 + \frac{\mu_j}{\phi_{1j}^2} & -\phi_{1j}\phi_{2j} \end{pmatrix}.$$

3. The Rosochatius deformed AKNS hierarchy with self-consistent sources

For the AKNS eigenvalue problem [40]

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_x = U \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \qquad U = \begin{pmatrix} -\lambda & q \\ r & \lambda \end{pmatrix},$$

and evolution equation of eigenfunction

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_{t_n} = V^{(n)} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \qquad V^{(n)} = \sum_{i=1}^{n} \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix} \lambda^{n-i},$$

the associated AKNS hierarchy reads

$$u_{t_n} = \begin{pmatrix} q \\ r \end{pmatrix}_{t_n} = J \begin{pmatrix} c_{n+1} \\ b_{n+1} \end{pmatrix} = J \frac{\delta H_{n+1}}{\delta u}$$

where

$$a_0 = -1, \qquad b_0 = c_0 = 0, \qquad a_1 = 0, \qquad b_1 = q, \qquad c_1 = r, \ldots,$$

$$\begin{pmatrix} c_{n+1} \\ b_{n+1} \end{pmatrix} = L^n \begin{pmatrix} r \\ q \end{pmatrix}, \qquad L = \frac{1}{2} \begin{pmatrix} \partial - 2r\partial^{-1}q & 2r\partial^{-1}r \\ -2q\partial^{-1}q & -\partial + 2q\partial^{-1}r \end{pmatrix},$$

$$a_{n,x} = qc_n - rb_n, \qquad H_n = \frac{2}{n}a_{n+1}, \qquad J = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix};$$

we have

$$\frac{\delta\lambda}{\delta q} = \frac{1}{2}\phi_2^2, \qquad \frac{\delta\lambda}{\delta r} = -\frac{1}{2}\phi_1^2.$$

The higher-order constrained flows of the AKNS hierarchy are given by [26, 27]

$$\frac{\delta H_{n+1}}{\delta u} - \frac{1}{2}\sum_{j=1}^{N} \frac{\delta\lambda_j}{\delta u} = \begin{pmatrix} c_{n+1} \\ b_{n+1} \end{pmatrix} - \frac{1}{4} \begin{pmatrix} \langle\Phi_2, \Phi_2\rangle \\ -\langle\Phi_1, \Phi_1\rangle \end{pmatrix} = 0, \tag{36a}$$

$$\phi_{1jx} = -\lambda_j\phi_{1j} + q\phi_{2j}, \qquad \phi_{2jx} = \lambda_j\phi_{2j} + r\phi_{1j}, \qquad j = 1, 2, \ldots, N. \tag{36b}$$

Equations (36) for $n = 2$ give the first higher-order constrained flow

$$-q_{xx} + 2q^2r - \sum_{j=1}^{N}\phi_{1j}^2 = 0, \qquad r_{xx} - 2qr^2 - \sum_{j=1}^{N}\phi_{2j}^2 = 0, \tag{37a}$$

$$\phi_{1jx} = -\lambda_j\phi_{1j} + q\phi_{2j}, \qquad \phi_{2jx} = \lambda_j\phi_{2j} + r\phi_{1j}, \qquad j = 1, 2, \ldots, N. \tag{37b}$$

Let $q_1 = q, q_2 = r, p_1 = -\frac{1}{2}r_x, p_2 = -\frac{1}{2}q_x, Q = (\phi_{11}, \phi_{12}, \cdots, \phi_{1N}, q_1, q_2)^T$ and $P = (\phi_{21}, \phi_{22}, \ldots, \phi_{2N}, p_1, p_2)^T$. Equations (37) become a FDIHS (19) with

$$H = -\langle\Lambda\Phi_1, \Phi_2\rangle + \frac{q_1}{2}\langle\Phi_2, \Phi_2\rangle - \frac{q_2}{2}\langle\Phi_1, \Phi_1\rangle + \frac{1}{2}q_1^2q_2^2 - 2p_1p_2,$$

and has the Lax representation (4) with the entries of Lax matrix $N^{(2)}$ being given by

$$A(\lambda) = -2\lambda^2 + q_1q_2 + \frac{1}{2}\sum_{j=1}^{N}\frac{\phi_{1j}\phi_{2j}}{\lambda-\lambda_j}, \qquad B(\lambda) = 2\lambda q_1 + 2p_2 - \frac{1}{2}\sum_{j=1}^{N}\frac{\phi_{1j}^2}{\lambda-\lambda_j},$$

$$C(\lambda) = 2\lambda q_2 - 2p_1 + \frac{1}{2}\sum_{j=1}^{N}\frac{\phi_{2j}^2}{\lambda-\lambda_j}.$$

With respect to the standard Poisson bracket it is found that

$$\begin{aligned}
&\{A(\lambda), A(\mu)\} = \{B(\lambda), B(\mu)\} = \{C(\lambda), C(\mu)\} = 0,\\
&\{A(\lambda), B(\mu)\} = \frac{B(\lambda)-B(\mu)}{\lambda-\mu},\\
&\{A(\lambda), C(\mu)\} = \frac{C(\mu)-C(\lambda)}{\lambda-\mu}, \qquad \{B(\lambda), C(\mu)\} = \frac{2[A(\lambda)-A(\mu)]}{\lambda-\mu},
\end{aligned} \tag{38}$$

which gives rise to (22).

Now we consider the Rosochatius deformation $\tilde{N}^{(2)}$ of the Lax matrix $N^{(2)}$

$$\tilde{A}(\lambda) = A(\lambda), \qquad \tilde{B}(\lambda) = B(\lambda), \qquad \tilde{C}(\lambda) = C(\lambda) + \frac{1}{2}\sum_{j=1}^{N}\frac{\mu_j}{(\lambda-\lambda_j)\phi_{1j}^2}. \tag{39}$$

It is easy to find that the elements in $\tilde{N}^{(2)}$ still keep the relations of the Poisson brackets (38) and (22).

A direct calculation gives

$$\tilde{A}^2(\lambda) + \tilde{B}(\lambda)\tilde{C}(\lambda) = 4\lambda^4 + P_0\lambda + P_1 + \sum_{j=1}^{N}\frac{P_j}{\lambda-\lambda_j} - \frac{1}{4}\sum_{j=1}^{N}\frac{\mu_j}{(\lambda-\lambda_j)^2} \tag{40}$$

where

$$\begin{aligned}
P_0 &= -4q_1p_1 + 4q_2p_2 - 2\langle\Phi_1, \Phi_2\rangle,\\
P_1 &= -4p_1p_2 + q_1^2q_2^2 + q_1\left(\langle\Phi_2, \Phi_2\rangle + \sum_{j=1}^{N}\frac{\mu_j}{\phi_{1j}^2}\right) - q_2\langle\Phi_1, \Phi_1\rangle - 2\langle\Lambda\Phi_1, \Phi_2\rangle,\\
P_{j+1} &= (\lambda_jq_1 + p_2)\left(\phi_{2j}^2 + \frac{\mu_j}{\phi_{1j}^2}\right) - (\lambda_jq_2 - p_1)\phi_{1j}^2 + \left(q_1q_2 - 2\lambda_j^2\right)\phi_{1j}\phi_{2j}\\
&\quad + \frac{1}{4}\sum_{k\neq j}\frac{1}{\lambda_j-\lambda_k}\left[2\phi_{1j}\phi_{2j}\phi_{1k}\phi_{2k} - \phi_{1j}^2\left(\phi_{2k}^2 + \frac{\mu_k}{\phi_{1k}^2}\right) - \phi_{1k}^2\left(\phi_{2j}^2 + \frac{\mu_j}{\phi_{1j}^2}\right)\right],\\
&\quad j = 1, \ldots, N.
\end{aligned} \tag{41}$$

Choosing $\frac{1}{2}P_1 = \tilde{H}$ as a Hamiltonian function, we get the following Hamiltonian system:

$$q_{1x} = -2p_2, \qquad q_{2x} = -2p_1, \tag{42a}$$

$$p_{1x} = -\frac{1}{2}\langle\Phi_2, \Phi_2\rangle - \sum_{j=1}^{N}\frac{\mu_j}{2\phi_{1j}^3} - q_2^2q_1, \qquad p_{2x} = \frac{1}{2}\langle\Phi_1, \Phi_1\rangle - q_1^2q_2, \tag{42b}$$

$$\phi_{1jx} = -\lambda_j\phi_{1j} + q_1\phi_{2j}, \qquad \phi_{2jx} = \lambda_j\phi_{2j} + q_2\phi_{1j} + \frac{\mu_jq_1}{\phi_{1j}^3}, \qquad j = 1, \ldots, N, \tag{42c}$$

which has the Lax representation (4) with $N^{(2)}$ replaced by $\tilde{N}^{(2)}$. Then (22) and (24) imply that $P_0, P_1, \ldots, P_{N+1}$ are $N+2$ independent first integrals in involution, so (42) is a FDIHS [41].

In general, in the similar way as in section 2, (36) becomes a FDIHS (19) with

$$H = \sum_{i=1}^{n} q_{i,x} p_i - H_{n+1} - \langle \Lambda\Phi_1, \Phi_2\rangle + \frac{q_1}{2}\langle \Phi_2, \Phi_2\rangle - \frac{q_2}{2}\langle \Phi_1, \Phi_1\rangle.$$

Then the integrable Rosochatius deformation of the higher-order constrained flow (36) is generated by the following Hamiltonian function:

$$\tilde{H} = \sum_{i=1}^{n} q_{i,x} p_i - H_{n+1} - \langle \Lambda\Phi_1, \Phi_2\rangle + \frac{q_1}{2}\langle \Phi_2, \Phi_2\rangle - \frac{q_2}{2}\langle \Phi_1, \Phi_1\rangle + \frac{1}{2}\sum_{j=1}^{N} \frac{\mu_j q_1}{\phi_{1j}^2}. \tag{43}$$

The AKNS hierarchy with self-consistent sources is [26]

$$\begin{pmatrix} q \\ r \end{pmatrix}_{t_n} = J\left[\frac{\delta H_{n+1}}{\delta u} - \frac{1}{2}\sum_{j=1}^{N}\frac{\delta\lambda_j}{\delta u}\right] = J\left[\begin{pmatrix} c_{n+1} \\ b_{n+1}\end{pmatrix} - \frac{1}{4}\begin{pmatrix} \langle\Phi_2, \Phi_2\rangle \\ -\langle \Phi_1, \Phi_1\rangle\end{pmatrix}\right], \tag{44a}$$

$$\phi_{1jx} = -\lambda_j\phi_{1j} + q\phi_{2j}, \qquad \phi_{2jx} = \lambda_j\phi_{2j} + r\phi_{1j}, \qquad j = 1, 2, \ldots, N. \tag{44b}$$

When $n = 2$, the AKNS equation with self-consistent sources reads [26]

$$2q_t = -q_{xx} + 2q^2 r - \sum_{j=1}^{N}\phi_{1j}^2, \qquad 2r_t = r_{xx} - 2qr^2 - \sum_{j=1}^{N}\phi_{2j}^2, \tag{45a}$$

$$\phi_{1jx} = -\lambda_j\phi_{1j} + q\phi_{2j}, \qquad \phi_{2jx} = \lambda_j\phi_{2j} + r\phi_{1j}, \qquad j = 1, \ldots, N. \tag{45b}$$

Based on (42), we obtain the integrable Rosochatius deformation of the AKNS equation with self-consistent sources

$$2q_t = -q_{xx} + 2q^2 r - \sum_{j=1}^{N}\phi_{1j}^2, \qquad 2r_t = r_{xx} - 2qr^2 - \sum_{j=1}^{N}\phi_{2j}^2 - \sum_{j=1}^{N}\frac{\mu_j}{\phi_{1j}^2}, \tag{46a}$$

$$\phi_{1jx} = -\lambda_j\phi_{1j} + q\phi_{2j}, \qquad \phi_{2jx} = \lambda_j\phi_{2j} + r\phi_{1j} + \frac{\mu_j q}{\phi_{1j}^3}, \qquad j = 1, \ldots, N \tag{46b}$$

which has the zero-curvature representation (6) with $N^{(2)}$:

$$N^{(2)} = \begin{pmatrix} -2\lambda^2 + qr & 2\lambda q - q_x \\ 2\lambda r + r_x & 2\lambda^2 - qr\end{pmatrix} + \frac{1}{2}\sum_{j=1}^{N}\frac{1}{\lambda - \lambda_j}\begin{pmatrix} \phi_{1j}\phi_{2j} & -\phi_{1j}^2 \\ \phi_{2j}^2 + \frac{\mu_j}{\phi_{1j}^2} & -\phi_{1j}\phi_{2j}\end{pmatrix}. \tag{47}$$

In general the integrable Rosochatius deformation of the AKNS hierarchy with self-consistent sources is given by

$$\begin{pmatrix} q \\ r \end{pmatrix}_{t_n} = J\left[\begin{pmatrix} c_{n+1} \\ b_{n+1}\end{pmatrix} - \frac{1}{4}\begin{pmatrix} \langle\Phi_2, \Phi_2\rangle + \sum_{j=1}^{N}\frac{\mu_j}{\phi_{1j}^2} \\ -\langle \Phi_1, \Phi_1\rangle\end{pmatrix}\right], \tag{48a}$$

$$\phi_{1jx} = -\lambda_j\phi_{1j} + q\phi_{2j}, \qquad \phi_{2jx} = \lambda_j\phi_{2j} + r\phi_{1j} + \frac{\mu_j q}{\phi_{1j}^3}, \qquad j = 1, 2, \ldots, N, \tag{48b}$$

which has the zero-curvature representation (6) with $N^{(n)}$ given by

$$N^{(n)} = V^{(n)} + \frac{1}{2}\sum_{j=1}^{N}\frac{1}{\lambda-\lambda_j}\begin{pmatrix}\phi_{1j}\phi_{2j} & -\phi_{1j}^2\\ \phi_{2j}^2+\frac{\mu_j}{\phi_{1j}^2} & -\phi_{1j}\phi_{2j}\end{pmatrix}.$$

Remark 3. In contrast to the Rosochatius deformation of KdV hierarchy with self-consistent sources, the Rosochatius deformation of AKNS hierarchy with self-consistent sources has the deformed term in both (48*a*) and (48*b*).

4. The Rosochatius deformed mKdV hierarchy with self-consistent sources

For the mKdV spectral problem [32]

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_x = U\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \qquad U=\begin{pmatrix}-u & \lambda\\ \lambda & u\end{pmatrix},$$

and evolution equation of eigenfunction

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_{t_n} = V^{(n)}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \qquad V^{(n)}=\sum_{i=1}^{n-1}\begin{pmatrix}a_i\lambda & b_i\\ c_i & -a_i\lambda\end{pmatrix}\lambda^{2n-2i-3}+\begin{pmatrix}a_n & 0\\ 0 & -a_n\end{pmatrix},$$

the associated mKdV hierarchy reads

$$u_{t_n} = -\partial a_n = \partial\frac{\delta H_n}{\delta u},$$

where

$$a_0=0, \qquad b_0=c_0=1, \qquad a_1=-u, \qquad b_1=-\frac{u^2}{2}+\frac{u_x}{2},$$

$$c_1=-\frac{u^2}{2}-\frac{u_x}{2},\ldots,$$

$$a_{n+1}=La_n, \qquad L=\frac{1}{4}\partial^2-u\partial^{-1}u\partial$$

$$b_n=\partial^{-1}u\partial a_n-\frac{1}{2}a_{nx}, \qquad c_n=\partial^{-1}u\partial a_n+\frac{1}{2}a_{nx};$$

we have

$$\frac{\delta\lambda}{\delta u}=\frac{1}{2}\phi_1\phi_2.$$

The higher-order constrained flows of the mKdV hierarchy are

$$\frac{\delta H_n}{\delta u}+2\sum_{j=1}^{N}\frac{\delta\lambda_j}{\delta u}\equiv -a_n+\sum_{j=1}^{N}\phi_{1j}\phi_{2j}=0, \tag{49a}$$

$$\phi_{1j,x}=-u\phi_{1j}+\lambda_j\phi_{2j}, \qquad \phi_{2j,x}=\lambda_j\phi_{1j}+u\phi_{2j}, \qquad j=1,2,\ldots,N. \tag{49b}$$

When $n=2$, (49) gives the first higher-order constrained flow

$$u_{xx}-2u^3=-4\sum_{j=1}^{N}\phi_{1j}\phi_{2j}=-4\langle\Phi_1,\Phi_2\rangle, \tag{50a}$$

$$\phi_{1jx} = \lambda_j\phi_{2j} - u\phi_{1j}, \qquad \phi_{2jx} = \lambda_j\phi_{1j} + u\phi_{2j}, \qquad j = 1, \ldots, N. \tag{50b}$$

Let $q_1 = u$, $p_1 = -\frac{u_x}{4}$, then it becomes a FDIHS (19) with

$$H = -q_1\langle\Phi_1, \Phi_2\rangle + \tfrac{1}{2}\langle\Lambda\Phi_2, \Phi_2\rangle - \tfrac{1}{2}\langle\Lambda\Phi_1, \Phi_1\rangle - 2p_1^2 + \tfrac{1}{8}q_1^4,$$

and has the Lax representation (4) with the entries of Lax matrix $N^{(2)}$ being given by

$$A(\lambda) = -q_1\lambda + \sum_{j=1}^{N}\frac{\lambda\phi_{1j}\phi_{2j}}{\lambda^2-\lambda_j^2}, \qquad B(\lambda) = \lambda^2 - \frac{1}{2}q_1^2 - 2p_1 - \sum_{j=1}^{N}\frac{\lambda_j\phi_{1j}^2}{\lambda^2-\lambda_j^2},$$

$$C(\lambda) = \lambda^2 - \frac{1}{2}q_1^2 + 2p_1 + \sum_{j=1}^{N}\frac{\lambda_j\phi_{2j}^2}{\lambda^2-\lambda_j^2}. \tag{51}$$

With respect to the standard Poisson bracket, a direct calculation gives

$$\begin{aligned}
&\{A(\lambda), A(\mu)\} = \{B(\lambda), B(\mu)\} = \{C(\lambda), C(\mu)\} = 0,\\
&\{A(\lambda), B(\mu)\} = 2\lambda\frac{B(\lambda)-B(\mu)}{\lambda^2-\mu^2},\\
&\{A(\lambda), C(\mu)\} = 2\lambda\frac{C(\lambda)-C(\mu)}{\mu^2-\lambda^2}, \qquad \{B(\lambda), C(\mu)\} = \frac{4[A(\mu)\mu - A(\lambda)\lambda]}{\mu^2-\lambda^2}
\end{aligned} \tag{52}$$

which leads to (22).

Now we consider the integrable Rosochatius deformation $\tilde{N}^{(2)}$ of the Lax matrix $N^{(2)}$

$$\tilde{A}(\lambda) = A(\lambda), \qquad \tilde{B}(\lambda) = B(\lambda), \qquad \tilde{C}(\lambda) = C(\lambda) + \sum_{j=1}^{N}\frac{\mu_j\lambda_j}{(\lambda^2-\lambda_j^2)\phi_{1j}^2}. \tag{53}$$

It is not difficult to find that the elements in $\tilde{N^{(2)}}$ still keep the relations of the Poisson brackets (52) and (22).

A direct calculation gives

$$\tilde{A}^2(\lambda) + \tilde{B}(\lambda)\tilde{C}(\lambda) = \lambda^4 + P_0 + \sum_{j=1}^{N}\frac{P_j}{\lambda^2-\lambda_j^2} - \sum_{j=1}^{N}\frac{\lambda_j\mu_j}{(\lambda^2-\lambda_j^2)^2} \tag{54}$$

where

$$\begin{aligned}
P_0 &= -2q_1\langle\Phi_1, \Phi_2\rangle + \langle\Lambda\Phi_2, \Phi_2\rangle - \langle\Lambda\Phi_1, \Phi_1\rangle + \sum_{j=1}^{N}\frac{\lambda_j\mu_j}{\phi_{1j}^2} + \frac{1}{4}q_1^4 - 4p_1^2\\
P_j &= -2q_1\lambda_j^2\phi_{1j}\phi_{2j} + \lambda_j^3\left(\phi_{2j}^2 + \frac{\mu_j}{\phi_{1j}^2}\right) - \left(\frac{1}{2}q_1^2 + 2p_1\right)\left(\lambda_j\phi_{2j}^2 + \frac{\lambda_j\mu_j}{\phi_{1j}^2}\right)\\
&\quad - \lambda_j^3\phi_{1j}^2 + \left(\frac{1}{2}q_1^2 - 2p_1\right)\lambda_j\phi_{1j}^2 - \sum_{k\neq j}\frac{1}{\lambda_j^2-\lambda_k^2}\left[2\lambda_j^2\phi_{1j}\phi_{2j}\phi_{1k}\phi_{2k}\right.\\
&\quad \left. + \lambda_j\lambda_k\phi_{1j}^2\left(\phi_{2k}^2 + \frac{\mu_k}{\phi_{1k}^2}\right) + \lambda_j\lambda_k\phi_{1k}^2\left(\phi_{2j}^2 + \frac{\mu_j}{\phi_{1j}^2}\right)\right].
\end{aligned} \tag{55}$$

Choosing $\frac{1}{2}P_0 = \tilde{H}$ as a Hamiltonian function, we get the following Hamiltonian system

$$q_{1x} = -4p_1, \qquad p_{1x} = \langle\Phi_1, \Phi_2\rangle - \frac{1}{2}q_1^3, \tag{56a}$$

$$\phi_{1jx} = \lambda_j\phi_{2j} - q_1\phi_{1j}, \qquad \phi_{2jx} = \lambda_j\phi_{1j} + q_1\phi_{2j} + \frac{\lambda_j\mu_j}{\phi_{1j}^3}, \qquad j = 1, \ldots, N \tag{56b}$$

which has the Lax representation (4) with $N^{(2)}$ replaced by $\tilde{N}^{(2)}$ and is a FDIHS.

Similarly, (49) can be transformed into a FDIHS (19) with $N+1$ independent first integrals $P_0, P_1, \ldots, P_N$ in involution,

$$H = \sum_{i=1}^{n-1} q_{i,x}p_i - H_n - q_1\langle\Phi_1, \Phi_2\rangle + \frac{1}{2}\langle\Lambda\Phi_2, \Phi_2\rangle - \frac{1}{2}\langle\Lambda\Phi_1, \Phi_1\rangle.$$

Then the integrable Rosochatius deformation of the higher-order constrained flow (49) is generated by the following Hamiltonian function:

$$\tilde{H} = \sum_{i=1}^{n-1} q_{i,x}p_i - H_n - q_1\langle\Phi_1, \Phi_2\rangle + \frac{1}{2}\langle\Lambda\Phi_2, \Phi_2\rangle - \frac{1}{2}\langle\Lambda\Phi_1, \Phi_1\rangle + \frac{1}{2}\sum_{j=1}^{N}\frac{\lambda_j\mu_j}{\phi_{1j}^2}. \tag{57}$$

The mKdV hierarchy with self-consistent sources is

$$u_{t_n} = \partial\left[\frac{\delta H_n}{\delta u} + 2\sum_{j=1}^{N}\frac{\delta\lambda_j}{\delta u}\right] \equiv \partial\left[-a_n + \sum_{j=1}^{N}\phi_{1j}\phi_{2j}\right], \tag{58a}$$

$$\phi_{1j,x} = -u\phi_{1j} + \lambda_j\phi_{2j}, \qquad \phi_{2j,x} = \lambda_j\phi_{1j} + u\phi_{2j}, \qquad j = 1, 2, \ldots, N. \tag{58b}$$

When $n = 2$, (58) gives the mKdV equation with self-consistent sources

$$u_t = \frac{u_{xxx}}{4} - \frac{3}{2}u^2u_x + \sum_{j=1}^{N}(\phi_{1j}\phi_{2j})_x, \tag{59a}$$

$$\phi_{1jx} = \lambda_j\phi_{2j} - u\phi_{1j}, \qquad \phi_{2jx} = \lambda_j\phi_{1j} + u\phi_{2j}, \qquad j = 1, \ldots, N. \tag{59b}$$

Based on (56), the integrable Rosochatius deformation of the mKdV equation with self-consistent sources is given by

$$u_t = \frac{u_{xx}}{4} - \frac{3}{2}u^2u_x + \sum_{j=1}^{N}(\phi_{1j}\phi_{2j})_x, \tag{60a}$$

$$\phi_{1jx} = \lambda_j\phi_{2j} - u\phi_{1j}, \qquad \phi_{2jx} = \lambda_j\phi_{1j} + u\phi_{2j} + \frac{\lambda_j\mu_j}{\phi_{1j}^3}, \qquad j = 1, \ldots, N. \tag{60b}$$

which has the zero-curvature representation (6) with $N^{(2)}$ given by

$$N^{(2)} = \begin{pmatrix} -u\lambda^2 - \frac{u_{xx}}{4} + \frac{u^3}{2} & \lambda^3 - \left(\frac{u^2}{2} - \frac{u_x}{2}\right)\lambda \\ \lambda^3 - \left(\frac{u^2}{2} + \frac{u_x}{2}\right)\lambda & u\lambda^2 + \frac{u_{xx}}{4} - \frac{u^3}{2} \end{pmatrix} + \begin{pmatrix} -\sum_{j=1}^N\phi_{1j}\phi_{2j} & 0 \\ 0 & \sum_{j=1}^N\phi_{1j}\phi_{2j} \end{pmatrix}$$
$$+ \sum_{j=1}^{N}\frac{\lambda}{\lambda^2 - \lambda_j^2}\begin{pmatrix} \lambda\phi_{1j}\phi_{2j} & -\lambda_j\phi_{1j}^2 \\ \lambda_j\left(\phi_{2j}^2 + \frac{\mu_j}{\phi_{1j}^2}\right) & -\lambda\phi_{1j}\phi_{2j} \end{pmatrix}. \tag{61}$$

In general, the integrable Rosochatius deformation of the mKdV hierarchy with self-consistent sources is given by

$$u_{t_n} = \partial\left[\frac{\delta H_n}{\delta u} + 2\sum_{j=1}^{N}\frac{\delta\lambda_j}{\delta u}\right] = \partial\left[-a_n + \sum_{j=1}^{N}\phi_{1j}\phi_{2j}\right] \tag{62a}$$

$$\phi_{1jx} = \lambda_j\phi_{2j} - u\phi_{1j}, \qquad \phi_{2jx} = \lambda_j\phi_{1j} + u\phi_{2j} + \frac{\lambda_j\mu_j}{\phi_{1j}^3}, \qquad j = 1,\ldots,N. \tag{62b}$$

which has the zero-curvature representation (6) with $N^{(n)}$ given by

$$N^{(n)} = \sum_{i=1}^{n-1}\begin{pmatrix} a_i\lambda & b_i \\ c_i & -a_i\lambda\end{pmatrix}\lambda^{2n-2i-3} + \begin{pmatrix} a_n - \sum_{j=1}^{N}\phi_{1j}\phi_{2j} & 0 \\ 0 & a_n + \sum_{j=1}^{N}\phi_{1j}\phi_{2j}\end{pmatrix}$$
$$+ \sum_{j=1}^{N}\frac{\lambda}{\lambda^2-\lambda_j^2}\begin{pmatrix}\lambda\phi_{1j}\phi_{2j} & -\lambda_j\phi_{1j}^2 \\ \lambda_j\left(\phi_{2j}^2 + \frac{\mu_j}{\phi_{1j}^2}\right) & -\lambda\phi_{1j}\phi_{2j}\end{pmatrix}.$$

5. Conclusion

Rosochatius-type integrable systems have important physical applications. However, the studies on Rosochatius deformation are limited to a few finite-dimensional integrable Hamiltonian system (FDIHS) at present. The main purpose of this paper is to propose a systematic method for generalizing the Rosochatius deformation for FDIHS to the Rosochatius deformation for infinite-dimensional integrable equations. We first construct an infinite number of integrable Rosochatius deformations of FDIHSs obtained from higher-order constrained flows of some soliton hierarchies. Then, based on the Rosochatius deformed higher-order constrained flows, we establish the integrable Rosochatius deformations of some soliton hierarchies with self-consistent sources. The Rosochatius deformations of the KdV hierarchy with self-consistent sources, of the AKNS hierarchy with self-consistent sources and of the mKdV hierarchy with self-consistent sources, together with their Lax pairs are obtained. The approach presented here can be applied to other cases.

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